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ON STEADY MOTIONS OF AN ELASTIC ROD WITH APPLICATION TO CONTACT PROBLEMS

T. R. NORDENHOLZ and O. M. O'REILLY†
Department of Mechanical Engineering, University of California at Berkeley, Berkeley, CA 94720-1740, U.S.A.

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Abstract—Based on the directed theory of an elastic rod developed by Green, Naghdi and several of their co-workers, this paper presents a formulation which is convenient to use when considering steady motions of elastic rods. A properly invariant linear theory of this formulation is also developed and used to analyze several systems. These systems involve surface constraints and find application in various technologies where elastic rod-like bodies are present.

Dedicated to the memory of Paul M. Naghdi

1. INTRODUCTION

A common feature of band saws, tape drives, yarn twisters and satellite retrieval systems, is a rod-like elastic body in motion. The initiation and control of, and disturbances to, this motion involve various contact mechanisms. The considerable work on this topic may, for the present purposes, be divided into two categories. In the first category, the classical theory of elasticity is used (see, e.g., Gladwell (1980) and Johnson (1985) for further references). This theory has the advantage of being able to adequately model the contact conditions and tractions on the lateral surface of the body, and the disadvantage of a paucity of exact or analytical solutions. The second category primarily uses Bernoulli-Euler and Timoshenko rod-theories (see, e.g., Wickert and Mote (1988) for further references). Although the analysis is considerably simpler, it is well known that rod theories of this type are unable to accommodate contact conditions. As noted by Essenburg (1975),‡ this deficiency can be attributed to their inherent inability to model lateral deformation.

The purpose of the present paper is to present a formulation of a directed rod theory which accommodates lateral deformation and still retains the analytical advantages of rod theories. The particular directed rod theory used in this paper was developed in a series of works by A. E. Green, P. M. Naghdi and their co-workers (see the survey article by Naghdi (1982) for further references). Although most of these papers use a Lagrangian or material formulation of the theory, an Eulerian or spatial formulation was established recently by Green and Naghdi (1993). This directed rod theory has the significant advantage of being able to accommodate lateral contraction. Its abilities in this respect may be traced to the correspondence between this theory and the classical three-dimensional theory (see, e.g., Green, Naghdi and Wenner (1974a) and Green and Naghdi (1993)). As noted by Naghdi and Rubin (1984), it is also sufficiently general to encompass various constrained rod, such as Bernoulli-Euler and Timoshenko, and string theories which have appeared in the literature. With relation to static contact problems, it was used by Naghdi and Rubin (1989) to show how several rod theories predict contact forces when the lateral surface of a rod-like body is in contact with a rigid surface.

Because of the dependency of the constitutive equations of an elastic rod on its reference state, we are unable to use the Eulerian formulation of the theory for the problems of interest here. Furthermore, the Lagrangian formulation did not prove to be convenient. In this paper, a particular representation of the motion of a directed curve is exploited to

† To whom correspondence should be addressed.
‡ For further details on this point, see Naghdi and Rubin (1989) and Timoshenko (1956).

formulate the theory of Green *et al.* in a convenient manner.† When the motion of the rod-like body is steady, in a particular sense defined in Section 3, we may use a result of Nordenholz and O'Reilly (1995) to formulate the problem using this representation. For several cases of interest, the resulting formulation provides a set of ordinary differential equations. These equations may be solved to determine the motion of the rod-like body and the contact forces on its lateral surface. Based on the linear theory developed by Green, Naghdi and Wenner (1974b), and extensions to the work of Naghdi and Rubin (1989), we establish a linear theory in Section 4. The resulting theory is valid for an elastic rod which in its natural (reference) configuration is straight, homogenous, prismatic and orthotropic. This theory is rendered properly invariant under superposed rigid body motions by extending some recent work of O'Reilly (1995). For the purpose of relating the developments of this paper to others which have appeared in the literature, a brief discussion of constrained theories is also presented in Sections 2 and 4.

Various representative applications are discussed in the final sections of this paper. Their selection is intended to illustrate how the formulation presented in this paper may be used in problems involving surface constraints. The applications discussed include the motion of a rod over a rigid obstacle, and the motion of a rod through a set of rollers. In the latter example, the two cases of pure adhesion and frictionless slip of the rod and rollers are discussed.‡ The methods used in these applications illuminate an analogy to problems in constrained dynamical systems which is discussed in Section 7. With this analogy in mind, a future avenue of research is outlined which will enable the methods of this paper to be used on other applications.

We note that the contact conditions formulated, and contact forces assumed, are established using the correspondences of the directed rod theory with the three-dimensional theory where a Coulomb friction model is assumed. The constitutive equations for the directed rod we employ were previously obtained by Green *et al.* using comparisons with static solutions from the classical theory of elasticity. For static contact problems, qualitative agreement was obtained by Naghdi and Rubin (1989) with the classical theory. It would certainly be of interest to further compare our solutions to those from the classical theory. Unfortunately, we were unable to find the precise corresponding problems in the literature.

The summation convention for repeated lower case Latin and Greek indices is used throughout this paper. Lower case Latin indices take values of 1, 2, 3, while lower case Greek indices take values of 1, 2. There are several notation differences in the literature on the directed theory referenced in this paper. For convenience, the notations of Naghdi (1982) and Naghdi and Rubin (1984, 1989) will be adhered to.

2. GENERAL BACKGROUND AND PRELIMINARIES

We recall, from Naghdi (1982), the concept of a material curve $\mathscr{L}$ together with 2 deformable vector fields, which are known as directors, attached at each material point of the curve. The curve, which is embedded in Euclidean three-space, and its directors is known as a *Cosserat* or *directed* curve $\mathscr{R}$. To specify the kinematics of $\mathscr{R}$, let the material point of $\mathscr{L}$ be uniquely specified by the convected (Lagrangian) coordinate ξ. The position vector $\mathbf{r}$ of an arbitrary material point and the two directors $\mathbf{d}_\alpha$ associated with this material point in the present configuration are uniquely specified by vector valued functions of ξ and t. These functions define a motion of the directed curve: $\mathbf{r} = \mathbf{r}(\xi, t)$, $\mathbf{d}_\alpha = \mathbf{d}_\alpha(\xi, t)$. In addition, it is assumed that the scalar triple product $[\mathbf{d}_1 \mathbf{d}_2 \mathbf{d}_3] > 0$, where $\mathbf{d}_3 = \partial \mathbf{r}/\partial \xi$. The velocity of the material point and the director velocities are denoted by $\mathbf{v} = \dot{\mathbf{r}}$ and $\mathbf{w}_\alpha = \dot{\mathbf{d}}_\alpha$, respectively, where the superposed dot denotes the material time derivative. We recall that

† The representation we use is considered classical for inextensible rod and string theories (see, e.g., Cohen and Epstein (1994) and Wickert and Mote (1988)). It has also been used for extensible elastic strings by Healey and Papadopoulos (1990) and O'Reilly and Varadi (1995), among others.

‡ A related problem, using the three-dimensional theory of linear elasticity, was analyzed by Bentall and Johnson (1967, 1968).

the fixed reference configuration of the directed curve is specified by the functions $\mathbf{R} = \mathbf{R}(\xi)$ and $\mathbf{D}_\alpha = \mathbf{D}_\alpha(\xi)$, where $[\mathbf{D}_1\, \mathbf{D}_2\, \mathbf{D}_3] > 0$ and $\mathbf{D}_3 = \partial\mathbf{R}/\partial\xi$.

The Lagrangian formulation of the local field equations are, from Naghdi (1982),

$$\begin{aligned} \dot{\lambda} &= 0, \\ \frac{\partial \mathbf{n}}{\partial \xi} + \lambda \mathbf{f} &= \lambda(\dot{\mathbf{v}} + y^\alpha \dot{\mathbf{w}}_\alpha), \\ \frac{\partial \mathbf{m}^\alpha}{\partial \xi} + \lambda \mathbf{l}^\alpha - \mathbf{k}^\alpha &= \lambda(y^\alpha \dot{\mathbf{v}} + y^{\alpha\beta}\dot{\mathbf{w}}_\beta), \\ \mathbf{d}_3 \times \mathbf{n} + \mathbf{d}_\alpha \times \mathbf{k}^\alpha + \frac{\partial \mathbf{d}_\alpha}{\partial \xi} \times \mathbf{m}^\alpha &= \mathbf{0}. \end{aligned} \tag{1}$$

The conservation laws (1) are statements of conservation of mass, linear momentum, director momenta and moment of momentum, respectively. We note that $(1)_3$ represents two separate conservation laws for director momentum. In (1), $\mathbf{n} = \mathbf{n}(\xi, t)$ is the contact force, $\mathbf{k}^\alpha = \mathbf{k}^\alpha(\xi, t)$ are the intrinsic director forces, $\mathbf{m}^\alpha = \mathbf{m}^\alpha(\xi, t)$ are the director forces, $\mathbf{f} = \mathbf{f}(\xi, t)$ is the assigned force, $\mathbf{l}^\alpha = \mathbf{l}^\alpha(\xi, t)$ are the assigned director forces, $y^\alpha = y^\alpha(\xi)$ and $y^{\alpha\beta} = y^{\beta\alpha} = y^{\alpha\beta}(\xi)$ are inertia coefficients. Finally, $\lambda = \rho d_{33}^{1/2}$ where $\rho = \rho(\xi, t)$ is the mass density per unit length of the directed curve and $d_{33} = \mathbf{d}_3 \cdot \mathbf{d}_3$.

In this paper, the directed theory is used to model a three-dimensional rod-like body. It was shown, by Green and Naghdi (1970) and Green *et al.* (1974a), that the various quantities of the directed curve can be placed in correspondence with those of classical continuum mechanics. Letting θ^i $(i = 1, 2, 3)$ denote a set of convected coordinates for the rod-like body which is embedded in Euclidean three-space, the position vector $\mathbf{p}$ of a material point of the body is approximated by, from Naghdi (1982),

$$\mathbf{p} = \mathbf{p}(\theta^1, \theta^2, \theta^3 = \xi, t) = \mathbf{r}(\xi, t) + \theta^\alpha \mathbf{d}_\alpha(\xi, t). \tag{2}$$

It follows, from (2), that the material curve $\mathscr{L}$ is identified by the set of material points of the body where $\theta^\alpha = 0$. Related remarks apply to the lateral surface of the body. It is also convenient to recall that the inertia coefficients, y^α and $y^{\alpha\beta}$, and the mass density λ for a directed curve are specified by weighted integrals from the three-dimensional theory (see, e.g., Naghdi (1982)). In a similar manner, the fields, $\mathbf{n}$, $\mathbf{m}^\alpha$ and $\mathbf{k}^\alpha$, may be placed in correspondence to weighted integrals of the appropriate traction vector (see, e.g., Green *et al.* (1974a)). The related expressions for the assigned force $\mathbf{f}$ and assigned director forces $\mathbf{l}^\alpha$ both contain contributions due to the body force, and the traction vector on the lateral surface of the body (see, e.g., Naghdi (1982)).

Supplementary to the local field equations, the following conditions hold at a point $\xi = \gamma(t)$ where there is either a point source of one or more of the momenta, and or a propagating point of discontinuity in the fields,†

$$\begin{aligned} [\![\lambda]\!]\dot{\gamma} &= 0, \\ \mathbf{F} + [\![\mathbf{n}]\!] + [\![\lambda(\mathbf{v} + y^\beta \mathbf{w}_\beta)]\!]\dot{\gamma} &= \mathbf{0}, \\ \mathbf{L}^\alpha + [\![\mathbf{m}^\alpha]\!] + [\![\lambda(y^\alpha \mathbf{v} + y^{\alpha\beta}\mathbf{w}_\beta)]\!]\dot{\gamma} &= \mathbf{0}, \\ \mathbf{M} + [\![\mathbf{r} \times \mathbf{n} + \mathbf{d}_\alpha \times \mathbf{m}^\alpha]\!] + [\![\mathbf{r} \times \lambda(\mathbf{v} + y^\beta \mathbf{w}_\beta) + \mathbf{d}_\alpha \times \lambda(y^\alpha \mathbf{v} + y^{\alpha\beta}\mathbf{w}_\beta)]\!]\dot{\gamma} &= \mathbf{0}. \end{aligned} \tag{3}$$

In (3), we have used the standard notation $[\![f]\!] = f(\gamma_+) - f(\gamma_-)$, where

† Specific cases of these conditions may be found elsewhere in the literature: see, e.g., Antman and Liu (1979), Cohen and Whitman (1977) and Xiong and Hutton (1994).

$$\gamma_{\pm} = \lim_{\delta \to 0} \gamma(t) \pm \delta.$$

The terms $\mathbf{F}$, $\mathbf{L}^{\alpha}$ and $\mathbf{M}$ represent point sources of linear momentum, director momenta and moment of momentum, respectively:

$$\mathbf{F} = \lim_{\delta \to 0} \int_{\gamma-\delta}^{\gamma+\delta} \lambda \mathbf{f} \, d\xi, \mathbf{L}^{\alpha} = \lim_{\delta \to 0} \int_{\gamma-\delta}^{\gamma+\delta} \lambda \mathbf{l}^{\alpha} \, d\xi,$$

$$\mathbf{M} = \lim_{\delta \to 0} \int_{\gamma-\delta}^{\gamma+\delta} (\mathbf{r} \times \mathbf{f} + \mathbf{d}_{\alpha} \times \mathbf{l}^{\alpha}) \lambda \, d\xi. \tag{4}$$

The discontinuity conditions are established using a standard procedure from the integral forms of the balance laws. These integral forms may be found in Naghdi (1982). The procedure is discussed in Truesdell and Toupin (1960) and Marshall and Naghdi (1989). In particular, motivated by the three-dimensional considerations discussed earlier, it is assumed that $\mathbf{n}$, $\mathbf{m}^{\alpha}$, $\mathbf{k}^{\alpha}$ and the momenta terms are finite. It is also assumed that $\mathbf{d}_{\alpha}$ and $\mathbf{r}$ are continuous functions of ξ. We note that the latter continuity assumptions and (4) imply that $(3)_{2,3}$ may be used to reduce $(3)_4$ to an identity.†

The constitutive relations and constrained responses for the fields, $\mathbf{n}$, $\mathbf{m}^{\alpha}$ and $\mathbf{k}^{\alpha}$, are obtained after first postulating the existence of a strain energy per unit mass ψ (see, e.g., Green *et al.* (1974b) and Naghdi and Rubin (1984)):

$$\psi = \hat{\psi}(\gamma_{ik}, \kappa_{\alpha k}, \mathbf{D}_k, \mathbf{D}'_{\alpha}, \xi), \tag{5}$$

where the deformation measures, γ_{ik} and $\kappa_{\alpha i}$, are defined by

$$2\gamma_{ik} = 2\gamma_{ki} = \mathbf{d}_i \cdot \mathbf{d}_k - \mathbf{D}_i \cdot \mathbf{D}_k,$$

$$\kappa_{\alpha i} = \mathbf{d}_i \cdot \mathbf{d}'_{\alpha} - \mathbf{D}_i \cdot \mathbf{D}'_{\alpha}, \tag{6}$$

and $'$ denotes $\partial/\partial\xi$. It is further assumed that the motion of the directed curve is subject to R constraints, each of which have the form

$$\phi_S = \hat{\phi}_S(\gamma_{ik}, \kappa_{\alpha k}, \mathbf{D}_k, \mathbf{D}'_{\alpha}, \xi) = 0, S = 1, \ldots, R. \tag{7}$$

We also recall the expression for the mechanical power P:

$$P = \mathbf{n} \cdot \mathbf{v}' + \mathbf{k}^{\alpha} \cdot \mathbf{w}_{\alpha} + \mathbf{m}^{\alpha} \cdot \mathbf{w}'_{\alpha}. \tag{8}$$

With the assistance of the relation, from Naghdi (1982), $P = \lambda \dot{\psi}$, $(1)_4$, (5), (7) and (8), the desired relations are obtained using a procedure similar to that used by Naghdi and Rubin (1984):‡

$$\mathbf{n} = \frac{\partial \Phi}{\partial \gamma_{3k}} \mathbf{d}_k + \frac{\partial \Phi}{\partial \kappa_{\alpha 3}} \mathbf{d}'_{\alpha}, \quad \mathbf{m}^{\alpha} = \frac{\partial \Phi}{\partial \kappa_{\alpha k}} \mathbf{d}_k, \quad \mathbf{k}^{\alpha} = \frac{\partial \Phi}{\partial \gamma_{\alpha k}} \mathbf{d}_k + \frac{\partial \Phi}{\partial \kappa_{\alpha \beta}} \mathbf{d}'_{\beta}, \tag{9}$$

where

† The continuity of $\mathbf{r}$ and $\mathbf{d}_{\alpha}$ as functions of ξ follows from their use in describing material curves (cf. (2) and Naghdi and Rubin (1989)). Concerning $\mathbf{M}$, a related situation is discussed in Marshall and Naghdi (1989).

‡ The form of the relations (9) differs from that presented in Naghdi and Rubin (1984). The difference is attributable to the requirement that each of the R constraint functions ϕ_S are assumed to be invariant under superposed rigid body motions of the directed curve. Requirements of this form for directed theories were apparently first used by Green *et al.* (1970), and were also used by O'Reilly (1995). We note that all of the constraints discussed in Naghdi and Rubin (1984) satisfy this requirement.

$$\Phi = \lambda\psi + \sum_{S=1}^{R} p_S\phi_S, \tag{10}$$

and $p_S = p_S(\xi, t)$ are indeterminate Lagrange multipliers. We note that the relations (9) identically satisfy the moment of momentum balance law $(1)_4$. As a consequence, this balance law is not discussed in the sequel.

For future reference, we recall that under a superposed rigid body motion, the motion of a directed curve transforms as†

$$\mathbf{r}^+ = \mathbf{r}^+(\xi, t+a) = \mathbf{Q}(t)\mathbf{r}(\xi, t) + \mathbf{q}(t), \mathbf{d}_\alpha^+ = \mathbf{d}_\alpha^+(\xi, t+a) = \mathbf{Q}(t)\mathbf{d}_\alpha(\xi, t), \tag{11}$$

where $\mathbf{Q}$ is a proper orthogonal tensor-valued function of time, $\mathbf{q}$ is a vector-valued function of time and a is a constant. It is assumed that the fields, $\mathbf{n}$, $\mathbf{m}^\alpha$ and $\mathbf{k}^\alpha$, are objective (see, e.g., Green *et al.* (1974b)):

$$\mathbf{n}^+ = \mathbf{Qn}, \quad \mathbf{m}^{\alpha+} = \mathbf{Qm}^\alpha, \quad \mathbf{k}^{\alpha+} = \mathbf{Qk}^\alpha. \tag{12}$$

As the relations (9) are properly invariant under superposed rigid body motions, it follows that, from O'Reilly (1995),

$$p_S^+ = p_S, S = 1, \ldots, R. \tag{13}$$

In addition, we shall assume that the discontinuity conditions (3) are properly invariant under superposed rigid body motions. This implies that

$$\gamma^+ = \gamma, \mathbf{F}^+ = \mathbf{QF}, \quad \mathbf{L}^{\alpha+} = \mathbf{QL}^\alpha, \quad \mathbf{M}^+ = \mathbf{QM}. \tag{14}$$

Following O'Reilly (1995), the relations (11), (12), (13) and (14) are used to establish the corresponding quantities for an auxiliary motion of the directed curve which will be defined later.

3. A CONVENIENT FORMULATION FOR STEADILY MOVING RODS

Our primary interest in this paper is rod-like bodies which are undergoing a particular motion, which we refer to as steady. In the context of a directed curve, a particular class of steady motions may be defined by 3 conditions: (i) the material curve $\mathscr{L}$ moves along a fixed curve $\mathscr{C}$; (ii) $\mathbf{v}$ is a function of position along $\mathscr{C}$ only; and (iii) $\mathbf{d}_\alpha$ are functions of position along $\mathscr{C}$ only. These conditions are from Green and Laws (1968). It was proven by Nordenholz and O'Reilly (1995) that for such a motion it is necessary and sufficient that the functions $\mathbf{r}$ and $\mathbf{d}_\alpha$ have the representations

$$\mathbf{r} = \mathbf{r}(\xi, t) = \bar{\mathbf{r}}(\xi + ct), \mathbf{d}_i = \mathbf{d}_i(\xi, t) = \bar{\mathbf{d}}_i(\xi + ct), \tag{15}$$

where c is a constant. As will presently be shown, the constant c is determined by the particular steady motion and the coordinatization ξ.

Motivated by the discussion of the previous paragraph, we define ζ as

$$\zeta = \hat{\zeta}(\xi, t) = \xi + ct. \tag{16}$$

In addition, it is convenient to define an intermediate configuration of the directed curve:

† For further details on invariance considerations, the reader is referred to Green and Naghdi (1979a), Green *et al.* (1974b) and Naghdi (1972).

$$\mathbf{R}_t = \mathbf{R}_t(\zeta) = \mathbf{R}(\xi + ct), \mathbf{D}_{t\alpha} = \mathbf{D}_{t\alpha}(\zeta) = \mathbf{D}_\alpha(\xi + ct), \tag{17}$$

where the functions $\mathbf{R}$ and $\mathbf{D}_\alpha$ define the reference configuration of the directed curve. The material curve $\mathscr{L}$ in the intermediate configuration is parameterized by the coordinate ζ. Physically, it corresponds to a motion of the directed curve where, at successive instants of time, each particle occupies the location of a neighboring particle and the directors are deformed so as to correspond to those associated with the neighboring particle. It should be clear that this motion cannot, in general, be sustained in the absence of assigned force $\mathbf{f}$ and assigned director forces $\mathbf{l}^\alpha$. In other words, the intermediate configuration is not, in general, a natural configuration of $\mathscr{R}$.†

It is convenient to describe the field equations, discontinuity conditions, deformation measures, strain energy, constraint equations and constitutive relations in terms of the variables associated with the intermediate configuration. This can be effected in a standard manner using (16), i.e., given an arbitrary function f of ξ and t, we may write

$$f = f(\xi, t) = f(\zeta - ct, t) = \tilde{f}(\zeta, t), \frac{\partial f}{\partial \xi} = \frac{\partial \tilde{f}}{\partial \zeta}, \dot{f} = \frac{\partial f}{\partial t} = c\frac{\partial \tilde{f}}{\partial \zeta} + \frac{\partial \tilde{f}}{\partial t}. \tag{18}$$

Briefly, the non-trivial local field equations are

$$\begin{aligned} \frac{\partial \mathbf{n}}{\partial \zeta} + \lambda \mathbf{f} &= \lambda\left(c\frac{\partial \mathbf{v}}{\partial \zeta} + \frac{\partial \mathbf{v}}{\partial t} + y^\alpha\left(c\frac{\partial \mathbf{w}_\alpha}{\partial \zeta} + \frac{\partial \mathbf{w}_\alpha}{\partial t}\right)\right), \\ \frac{\partial \mathbf{m}^\alpha}{\partial \zeta} + \lambda \mathbf{l}^\alpha - \mathbf{k}^\alpha &= \lambda\left(y^\alpha\left(c\frac{\partial \mathbf{v}}{\partial \zeta} + \frac{\partial \mathbf{v}}{\partial t}\right) + y^{\alpha\beta}\left(c\frac{\partial \mathbf{w}_\beta}{\partial \zeta} + \frac{\partial \mathbf{w}_\beta}{\partial t}\right)\right), \end{aligned} \tag{19}$$

where the fields, $\mathbf{n}$, $\mathbf{m}^\alpha$ and $\mathbf{k}^\alpha$, kinematical variables, $\mathbf{d}_\alpha$, $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{w}_\alpha$, and coefficients λ, y^α and $y^{\alpha\beta}$, are henceforth understood to be functions of ζ and t. The non-trivial discontinuity conditions are obtained from (3) by changing the functional dependence using (16):

$$\begin{aligned} \mathbf{F} + [\![\mathbf{n}]\!] + [\![\lambda(\mathbf{v} + y^\beta \mathbf{w}_\beta)]\!](\dot{\mu} - c) &= \mathbf{0}, \\ \mathbf{L}^\alpha + [\![\mathbf{m}^\alpha]\!] + [\![\lambda(y^\alpha \mathbf{v} + y^{\alpha\beta}\mathbf{w}_\beta)]\!](\dot{\mu} - c) &= \mathbf{0}, \end{aligned} \tag{20}$$

where $\mu(t) = \gamma(t) + ct$. The relations for $\mathbf{n}$, $\mathbf{m}^\alpha$ and $\mathbf{k}^\alpha$ are also easily obtained from (5), (6), (7), (9), (10) and (16). In the interests of brevity, they are not recorded here.

A solution of the partial differential equations governing the directed curve necessitates the specification of an initial velocity $\mathbf{v}_0 = \mathbf{v}(\zeta_0, t = t_0)$ for some $\zeta = \zeta_0$. In the event that the motion is steady in the sense defined previously, it is easily seen from a relation of the form $(18)_3$ that $\mathbf{v}_0$ and the initial conditions $\mathbf{r}_0 = \mathbf{r}(\zeta, t = t_0)$ suffice to determine c. Furthermore, if the remaining quantities in (19) and (20) and the relevant boundary conditions are independent of t, then a determination of the steady motion involves the solution of a set of ordinary differential equations. The procedure for obtaining a solution to these equations is similar to that employed for a static problem.‡ It should be clear, from (15) and (16), that the coordinate ζ in this case parameterizes the fixed curve $\mathscr{C}$ in space along which the material curve $\mathscr{L}$ moves.

† For strings, an intermediate configuration which is a natural configuration was used by O'Reilly (1996b) and O'Reilly and Varadi (1995). Its use was motivated by the earlier work of Antman and Reeken (1986).

‡ In the context of strings this result has been exploited by various authors: Healey and Papadopoulos (1990), O'Reilly (1996b) and Routh (1905), among others. For inextensible rods it has been used by Cohen and Epstein (1994) and Coleman and Dill (1992), among others.

4. A PROPERLY INVARIANT LINEAR THEORY FOR STEADY MOTIONS

As a preliminary to establishing a properly invariant linear theory, we consider a motion of the directed curve. From Naghdi (1982), we recall the definition of the invertible tensor $\mathbf{F}_{\mathscr{M}}$

$$\mathbf{F}_{\mathscr{M}} = \mathbf{F}_{\mathscr{M}}(\xi, t) = \mathbf{d}_k \otimes \mathbf{D}^k. \tag{21}$$

The invertibility of $\mathbf{F}_{\mathscr{M}}$ implies the existence of a unique polar decomposition:

$$\mathbf{F}_{\mathscr{M}} = \mathbf{SU}, \tag{22}$$

where $\mathbf{S}$ is a proper orthogonal tensor and $\mathbf{U}$ is a symmetric positive definite tensor.

We now consider a generalization of the auxiliary motion discussed by O'Reilly (1995). We choose a point, known as the *pivot*, $\bar{\xi} = \bar{\xi}(t)$ of $\mathscr{L}$, and use it and (22) to define an *auxiliary motion* of $\mathscr{R}$:

$$\begin{aligned} \mathbf{r}^* &= \mathbf{r}^*(\xi, t^*) = \mathbf{S}^{\mathrm{T}}(\bar{\xi}, t)(\mathbf{r}(\xi, t) - \mathbf{r}(\bar{\xi}, t)) + \mathbf{g}^*(t), \\ \mathbf{d}^*_\alpha &= \mathbf{d}^*_\alpha(\xi, t^*) = \mathbf{S}^{\mathrm{T}}(\bar{\xi}, t)\mathbf{d}_\alpha(\xi, t), \\ t^* &= t + c^*, \end{aligned} \tag{23}$$

where c^* is a constant and $\mathbf{g}^*(t)$ is a vector-valued function of time only. The auxiliary motion discussed by O'Reilly (1995) is obtained, from (23), by selecting $\bar{\xi}$ = constant and $\mathbf{g}^*(t) = \mathbf{R}(\bar{\xi})$. The principal use of the auxiliary motion is to obtain a properly invariant theory for infinitesimal deformations of the directed curve.†

The following theorem follows directly from the corresponding results in O'Reilly (1995) with some minor modifications.

Theorem

For specific choices of $\bar{\xi} = \bar{\xi}(t)$ and $\mathbf{g}^*(t)$, two motions of $\mathscr{R}$ differ by a rigid motion if and only if their auxiliary motions are (modulo an additive time constant) equal.

Following O'Reilly (1995), rather than using the motion of $\mathscr{R}$, we will use its auxiliary motion. The resulting formulation is properly invariant under arbitrary superposed rigid body motions of the directed curve. Specifically, the fields, balance laws, constitutive relations and constraints for the auxiliary motion are obtained from those for the motion $\mathbf{r} = \mathbf{r}(\xi, t)$, $\mathbf{d}_\alpha = \mathbf{d}_\alpha(\xi, t)$, by noting that the auxiliary motion is a particular superposed rigid body motion of this motion (O'Reilly (1995)). In addition, (11), (12), (13) and (14) are used in a similar manner to establish the discontinuity conditions. Henceforth we shall employ the auxiliary motion exclusively and the asterisk accompanying all quantities associated with it will be dropped.

We now proceed to develop a linear theory which models an elastic rod-like body. This body in its natural reference state is assumed to be straight, prismatic, isotropic, homogenous and of constant orthotropic cross-sections. Motivated by the results of Section 3, we choose

$$\bar{\xi}(t) = \bar{\bar{\xi}} + ct, \mathbf{g}^*(t) = \mathbf{R}(\bar{\bar{\xi}} + ct), c^* = 0, \tag{24}$$

where $\bar{\bar{\xi}}$ is a constant. It is convenient to define a fixed Cartesian basis $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ for Euclidean three-space and to choose ξ and the directors $\mathbf{D}_\alpha$ such that

† In this context, it is appropriate to recall, from O'Reilly (1995), that although the constitutive relations (9) are invariant under superposed rigid body motions, their approximations in an infinitesimal theory are not. This deficiency is partially attributed to the lack of invariance of the approximations to the deformation measures (6) which are used in the linear theory.

$$\mathbf{R} = (\zeta - ct)\mathbf{E}_3, \mathbf{D}_\alpha = \mathbf{E}_\alpha. \tag{25}$$

With the assistance of (25), the motion of $\mathscr{R}$ can be represented as

$$\begin{aligned} \mathbf{r} &= \mathbf{r}(\zeta, t) = \zeta\mathbf{E}_3 + \mathbf{u}(\zeta, t), \\ \mathbf{d}_\alpha &= \mathbf{d}_\alpha(\zeta, t) = \boldsymbol{\delta}_\alpha(\zeta, t) + \mathbf{E}_\alpha, \end{aligned} \tag{26}$$

where $\mathbf{u}$ and $\boldsymbol{\delta}_\alpha$ are introduced as *relative* measures of this motion. A measure of the deformation induced by the motion of $\mathscr{R}$ is obtained from

$$\varepsilon(t) = \max_{\alpha=1,2}\left\{\sup_{\zeta\in\mathscr{R}} \|\boldsymbol{\delta}_\alpha\|, \sup_{\zeta\in\mathscr{R}}\left\|\frac{\partial\mathbf{u}}{\partial\zeta}\right\|, \sup_{\zeta\in\mathscr{R}}\left\|\bar{L}\frac{\partial\boldsymbol{\delta}_\alpha}{\partial\zeta}\right\|\right\}, \tag{27}$$

where $\bar{L}$ is a (constant) characteristic length of $\mathscr{L}$. The measure ε is similar to that used in the development of a linear theory by Green *et al.* (1967) and Green *et al.* (1974a,b), and the properly invariant infinitesimal theory by O'Reilly (1995).

The establishment of a linear theory for the motion of an initially straight directed curve now proceeds, with some minor modifications, as in sections 3 and 8 of Green *et al.* (1974a,b). Various constrained theories of this general theory can be incorporated using supplementary material from Naghdi and Rubin (1984, 1989) and O'Reilly (1995).† We note that the constitutive relations and constrained responses discussed by these authors will be identical to those for the properly invariant theory.‡ As in Green *et al.* (1974b), the general linear theory reduces to 4 sets of equations for this case.

For future reference, and to partially illuminate our previous comments, we record here the equations pertaining to extension and flexure in the $\mathbf{E}_1$ direction. These laws and relations hold at all points of the rod except at a point of discontinuity. They are obtained, with some modifications of the inertial terms, functional dependencies and notation, from eqns (9.5) and (9.6) of Green *et al.* (1974b). The balance laws pertaining to flexure are§

$$\begin{aligned} \frac{\partial n_1}{\partial\zeta} + \lambda f_1 &= \lambda\left(c^2\frac{\partial^2 u_1}{\partial\zeta^2} + 2c\frac{\partial^2 u_1}{\partial\zeta\partial t} + \frac{\partial^2 u_1}{\partial t^2}\right), \\ \frac{\partial m_{13}}{\partial\zeta} - k_{13} + \lambda l_{13} &= \lambda y^{11}\left(c^2\frac{\partial^2\delta_{13}}{\partial\zeta^2} + 2c\frac{\partial^2\delta_{13}}{\partial\zeta\partial t} + \frac{\partial^2\delta_{13}}{\partial t^2}\right), \end{aligned} \tag{28}$$

where k_{13}, n_1, and m_3 are prescribed by constitutive relations and the moment of momentum balance law:

$$k_{13} = n_1 = \alpha_6\left(\delta_{13} + \frac{\partial u_1}{\partial\zeta}\right), m_{13} = \alpha_{16}\frac{\partial\delta_{13}}{\partial\zeta}. \tag{29}$$

The following notational convention was used in writing (28) and (29):

$$\begin{aligned} n_i &= \mathbf{n}\cdot\mathbf{E}_i, m_{\alpha i} = \mathbf{m}^\alpha\cdot\mathbf{E}_i, k_{\alpha i} = \mathbf{k}^\alpha\cdot\mathbf{E}_i, \\ f_i &= \mathbf{f}\cdot\mathbf{E}_i, l_{\alpha i} = \mathbf{l}^\alpha\cdot\mathbf{E}_i, u_i = \mathbf{u}\cdot\mathbf{E}_i, \delta_{\alpha i} = \boldsymbol{\delta}_\alpha\cdot\mathbf{E}_i, \end{aligned} \tag{30}$$

† Because the applications discussed in the later sections are not suited to the use of constrained theories, these theories are not pursued any further.

‡ This identification is also used in other works on the properly invariant theory: Casey and Naghdi (1981, 1985), Naghdi and Vongsarnpigoon (1983, Sect. 4.2) and O'Reilly (1995, 1996a).

§ As in the classical theory of elasticity, rather than writing these equations as functions of (ζ, t) they may be easily written as functions of (z, t) where z is the spatial coordinate. To the order of approximation considered here, the resulting equation as functions of (z, t) will have the same form.

where the components of $\boldsymbol{\delta}_\alpha$ are ornamented with an overbar to avoid any confusion with the Kronecker delta. The balance laws pertaining to extension are

$$\frac{\partial n_3}{\partial \zeta} + \lambda f_3 = \lambda\left(c^2\frac{\partial^2 u_3}{\partial \zeta^2} + 2c\frac{\partial^2 u_3}{\partial \zeta \partial t} + \frac{\partial^2 u_3}{\partial t^2}\right),$$

$$\frac{\partial m_{11}}{\partial \zeta} + \lambda l_{11} - k_{11} = \lambda y^{11}\left(c^2\frac{\partial^2 \bar{\delta}_{11}}{\partial \zeta^2} + 2c\frac{\partial^2 \bar{\delta}_{11}}{\partial \zeta \partial t} + \frac{\partial^2 \bar{\delta}_{11}}{\partial t^2}\right),$$

$$\frac{\partial m_{22}}{\partial \zeta} + \lambda l_{22} - k_{22} = \lambda y^{22}\left(c^2\frac{\partial^2 \bar{\delta}_{22}}{\partial \zeta^2} + 2c\frac{\partial^2 \bar{\delta}_{22}}{\partial \zeta \partial t} + \frac{\partial^2 \bar{\delta}_{22}}{\partial t^2}\right), \tag{31}$$

where n_3, m_{11}, k_{11}, and k_{22} are prescribed by constitutive relations:†

$$m_{11} = \alpha_{10}\frac{\partial \bar{\delta}_{11}}{\partial \zeta} + \alpha_{17}\frac{\partial \bar{\delta}_{22}}{\partial \zeta}, m_{22} = \alpha_{17}\frac{\partial \bar{\delta}_{11}}{\partial \zeta} + \alpha_{11}\frac{\partial \bar{\delta}_{22}}{\partial \zeta},$$

$$k_{11} = \alpha_1\bar{\delta}_{11} + \alpha_7\bar{\delta}_{22} + \alpha_8\frac{\partial u_3}{\partial \zeta}, k_{22} = \alpha_7\bar{\delta}_{11} + \alpha_2\bar{\delta}_{22} + \alpha_9\frac{\partial u_3}{\partial \zeta},$$

$$n_3 = \alpha_8\bar{\delta}_{11} + \alpha_9\bar{\delta}_{22} + \alpha_3\frac{\partial u_3}{\partial \zeta}. \tag{32}$$

It is also convenient to recall, for rectangular rods of height $2h$ and width w, the coefficients of the constitutive relations:

$$\alpha_1 = \alpha_2 = \alpha_3 = \frac{EA(1-v)}{(1+v)(1-2v)}, \alpha_6 = \frac{5EA}{12(1+v)},$$

$$\alpha_7 = \alpha_8 = \alpha_9 = \frac{EAv}{(1+v)(1-2v)},$$

$$\alpha_{10} = \frac{E}{2(1+v)}I_2, \alpha_{11} = \frac{E}{2(1+v)}I_1, \alpha_{16} = EI_2, \alpha_{17} = 0, \tag{33}$$

where $A = 2hw$, $I_1 = w^3h/6$, $I_2 = 2h^3w/3$, E is Young's modulus and v is Poisson's ratio. Apart from α_6, α_{10}, α_{11} and α_{17}, the coefficients were obtained from Green *et al.* (1974b). Of the remaining four coefficients, α_6 was obtained from Naghdi and Rubin (1989), and α_{10}, α_{11} and α_{17} were obtained from Green and Naghdi (1979b).‡ It is appropriate to note that these coefficients were obtained using comparisons to exact static solutions of the classical theory of linear elasticity.§ For a rectangular rod of uniform mass density ρ_0 per unit volume, the remaining (non-trivial) coefficients are obtained using (16) and relations discussed in Green *et al.* (1974b) and Naghdi (1982):

$$\lambda = \rho_0 A, \lambda y^{11} = \rho_0\frac{2h^3w}{3} = \rho_0 I_2, \lambda y^{22} = \rho_0\frac{hw^3}{6} = \rho_0 I_1. \tag{34}$$

The assigned force and assigned director forces are obtained using (16) and identifications with classical continuum mechanics (see, e.g., Naghdi (1982)).

Concerning the properly invariant infinitesimal theory, we now recall a result from O'Reilly (1995) which states that the choice of pivot is immaterial to the order of approximation being considered here. In the present context, this implies that for two different

† The notation used here for the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_{17}$ follows Naghdi and Rubin (1989).

‡ These three coefficients are set to 0 in Naghdi and Rubin (1989). For the formulation being considered in this paper it is necessary to include these coefficients.

§ The reader is referred to Green and Naghdi (1990) for further comments on this point.

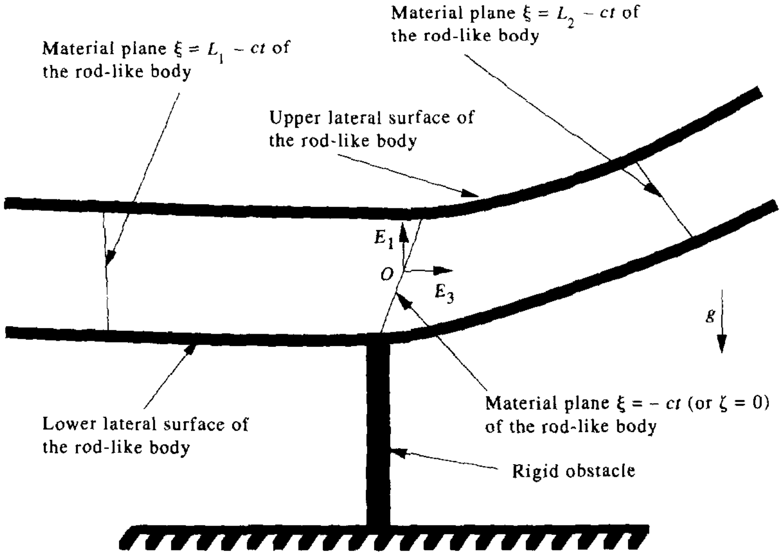

Fig. 1. A schematic diagram showing a rod which is moving over a rigid obstacle. The obstacle is idealized as providing a point contact with the lower lateral surface of the rod. The point **O** is such that $\mathbf{R}(\zeta = 0, t) = \mathbf{0}$, (cf. eqn (37)).

choices of $\bar{\xi}$ in $(24)_{1,2}$, the respective solutions of the equations governing the motion of the directed curve will be identical to $\mathbf{O}(\varepsilon)$ as $\varepsilon \to 0$. It follows that the extension to the invariant theory considered in this paper, namely permitting the pivot to vary as a prescribed function of time, does not affect the resulting infinitesimal theory. This result is further exploited in the subsequent applications by assuming that the rotation $\mathbf{S}$ associated with $\mathbf{F}_{\mathscr{R}}$ for some point $\bar{\xi}+ct$ is the identity. Furthermore, $\mathbf{g}^*(t)$ in $(24)_2$ is then chosen to correspond to the position vector of this point. It follows in this case that the auxiliary motion of $\mathscr{R}$ corresponds (trivially) to the actual motion of $\mathscr{R}$. Finally, the rod is assumed to have an infinite length in the applications being considered, but our attention is focused only on a finite length of ζ. As a consequence, the pivot does not necessarily have to lie in this finite region.

5. A ROD-LIKE BODY MOVING STEADILY OVER A RIGID OBSTACLE

Our interest lies in determining the deformation of a rod of rectangular cross section which is being drawn over a sharp rigid obstacle as shown in Fig. 1. The height and width of the rod are $2h$ and w, respectively. A gravitational force acts on the rod during this motion. The reference state of the rod is assumed to satisfy the conditions discussed prior to (24). The linear theory for this case again reduces to four sets of equations. Two of these sets govern the $(\mathbf{E}_2)$ flexural and torsional deformations of the rod, respectively (see Green *et al.* (1974b)). We first assume that the deformation of the directed curve satisfies

$$\mathbf{u}\cdot\mathbf{E}_2 = 0, \quad \boldsymbol{\delta}_1\cdot\mathbf{E}_2 = 0, \quad \boldsymbol{\delta}_2\cdot\mathbf{E}_1 = \boldsymbol{\delta}_2\cdot\mathbf{E}_3 = 0. \tag{35}$$

The assumptions (35) ensure that these two sets of equations are trivially satisfied. The remaining two sets describe extension and flexure in the $\mathbf{E}_1$ direction (see Green *et al.*, (1974b)).† It will become evident that these two sets of equations will be coupled due to the contact condition present in the problem of interest. In this section, we will indicate how the solution may be obtained. Several of the analytical details are placed in Appendix A in the interests of brevity and comprehension.

Apart from the value of ζ which corresponds to the rod contacting the obstacle, the assigned director forces, l_{11} and l_{22}, and the assigned forces, f_1 and f_3, are specified with the assistance of (34) and relations from Green *et al.* (1974a):

† A similar situation arose in the problem considered by Naghdi and Rubin (1989).

$$f_1 = -g, f_3 = 0, l_{11} = 0, l_{22} = 0. \tag{36}$$

It shall henceforth be assumed that the motion of the rod is steady in the sense defined in Section 3. Further, the material points of the three-dimensional rod-like body which are in contact with the obstacle are defined by the coordinates $\theta^1 = -h$, and, for convenience, $\theta^3 = \xi = -ct$ (cf. (2) and (16)). For these material points a contact condition is assumed to hold:

$$u_1(0) - h\bar{\delta}_{11}(0) = L. \tag{37}$$

In addition, at the contact point, $\mathbf{u}$ and $\boldsymbol{\delta}_\alpha$ are assumed to be continuous:†

$$[\![u_1]\!] = 0, [\![u_3]\!] = 0, \quad [\![\bar{\delta}_{13}]\!] = 0, [\![\bar{\delta}_{11}]\!] = 0, \quad [\![\bar{\delta}_{22}]\!] = 0. \tag{38}$$

Finally, as the contact point is a point of discontinuity of the assigned forces and assigned director forces, five conditions must be satisfied there:

$$\left[\!\left[(\alpha_6 - \lambda c^2)\frac{\mathrm{d}u_1}{\mathrm{d}\zeta} + \alpha_6\bar{\delta}_{13}\right]\!\right] = -\bar{q}, \quad \left[\!\left[(\alpha_{10} - \lambda y^{11}c^2)\frac{\mathrm{d}\bar{\delta}_{11}}{\mathrm{d}\zeta}\right]\!\right] = h\bar{q},$$

$$\left[\!\left[(\alpha_1 - \lambda c^2)\frac{\mathrm{d}u_3}{\mathrm{d}\zeta} + \alpha_7(\bar{\delta}_{11} + \bar{\delta}_{22})\right]\!\right] = 0,$$

$$\left[\!\left[(\alpha_{16} - \lambda y^{11}c^2)\frac{\mathrm{d}\bar{\delta}_{13}}{\mathrm{d}\zeta}\right]\!\right] = 0, \quad \left[\!\left[(\alpha_{11} - \lambda y^{22}c^2)\frac{\mathrm{d}\bar{\delta}_{22}}{\mathrm{d}\zeta}\right]\!\right] = 0. \tag{39}$$

These conditions are obtained from (20) with the added assistance of (29) and (32). The fields $\mathbf{F}$ and $\mathbf{L}^\alpha$ in (20) were specified by constitutive relations (cf. (4) and (16)):

$$\mathbf{F} = \bar{q}\mathbf{E}_1, \quad \mathbf{L}^1 = -h\bar{q}\mathbf{E}_1, \quad \mathbf{L}^2 = \mathbf{0}. \tag{40}$$

Specifically, we are assuming that the obstacle exerts a uniformly distributed uni-directional traction $(\bar{q}/w)\mathbf{E}_1$ on the lower lateral surface of the rod. Physically, this means that we are ignoring frictional effects.

It remains to specify eleven independent conditions. One of these, which is peculiar to problems on drawn rods, determines the constant c.‡ We shall presume in what follows that c is known. As a possible example of the remaining ten conditions, the following five quantities are specified at $\zeta = L_2$:

$$n_1(\zeta = L_2), n_3(\zeta = L_2), m_{11}(\zeta = L_2), m_{22}(\zeta = L_2), m_{13}(\zeta = L_2), \tag{41}$$

while at $\zeta = L_1$, the displacement fields are specified:

$$u_1(\zeta = L_1), u_3(\zeta = L_1), \bar{\delta}_{11}(\zeta = L_1), \bar{\delta}_{22}(\zeta = L_1), \bar{\delta}_{13}(\zeta = L_1). \tag{42}$$

Other examples of such conditions may easily be envisaged.

A solution of the drawn rod problem involves establishing the general solutions of (28), (29), (31)–(34) for $\mathbf{u}$ and $\boldsymbol{\delta}_\alpha$ over the two regions prior and subsequent to the obstacle. These solutions are obtained using standard methods from the theory of ordinary differential equations and are presented in the Appendix. In total, ten unknown constants will be present for both regions. An additional unknown is the contact force $\bar{q}$. The continuity and discontinuity conditions, (38) and (39), are then used to relate these constants, and

† Related conditions were recorded by Naghdi and Rubin (1989). As noted by them, these conditions are statements of the material nature of the lateral surfaces of the rod.

‡ The reader is referred to the discussion in the last paragraph of Section 3 for further details on this point.

reduce the number of unknowns to eleven. Finally, ten boundary conditions (e.g., (41) and (42)), and the contact condition (37) are enforced to obtain a complete solution to the boundary value problem. It is important to note that if the ten boundary conditions pertain solely to one region and satisfy the contact condition, then no explicit solution for the other region will be available. The reason for this lies in the inability of these conditions to determine $\bar{q}$. This summarizes the solution of the problem.

6. A ROD-LIKE BODY MOVING STEADILY BETWEEN TWO ROLLERS

Our interest lies in determining the deformation and contact forces for a rod-like body which is in motion over a single roller or between two rollers as shown in Fig. 2. In applications, the roller serves one of two distinct purposes: to control the displacement of a rod-like body or to act as an actuating device for the motion of the rod-like body. The criterion to distinguish these two functions is to determine the $\mathbf{E}_2$ component of the resultant external moment relative to the center of the roller from a balance of angular momentum. If the roller serves purely as a displacement controller, then, after neglecting viscous dissipation and the angular acceleration of the roller, this component should be zero. Otherwise, the roller is said to serve as an actuating device.

In this section, we will first formulate the contact conditions, contact forces and balance laws for a rod-like body moving over circular rollers. A directed curve is used to model the rod-like body and steady motions of this curve are assumed. The rollers are assumed to be rigid and have constant angular velocity. This is followed by a discussion of the coupling between the extensional and flexural deformations induced by the roller. We close this section with an example of the extensional deformation induced in a rod-like body when it is pulled or drawn between two rollers. This example serves as a qualitative comparison to the work of Bentall and Johnson (1967, 1968) who considered a similar problem, using a different method, for an elastic strip. In what follows, a directed curve is used to model the rod-like body, and steady motions of this curve and the rollers are assumed. For ease of exposition, several of the analytical details are placed in Appendix B.

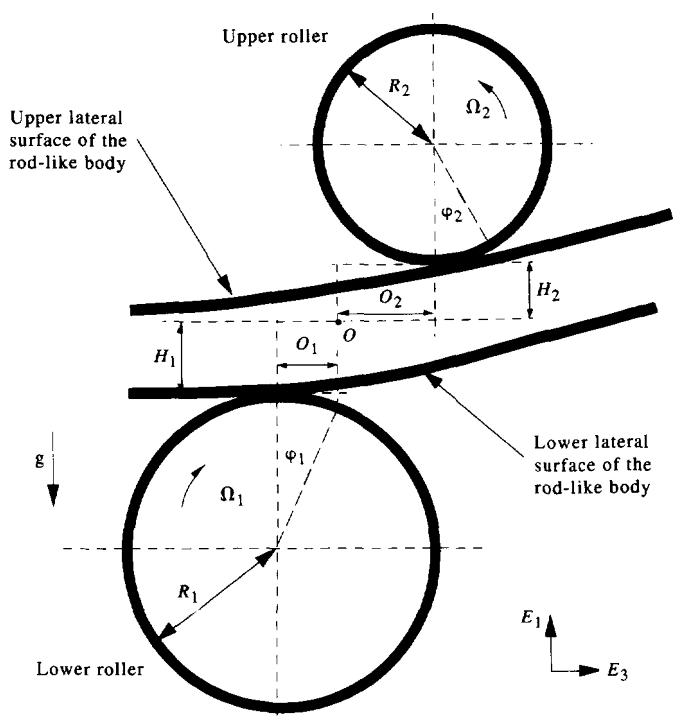

Fig. 2. A schematic diagram showing a rod which is being drawn between two rollers. The rollers may serve as either guides or pulleys. The point $\mathbf{O}$ is such that $\mathbf{R}(\zeta = 0, t) = \mathbf{0}$; it is referred to as the origin in Section 6.

6.1. *Contact conditions, contact forces and governing equations*

With regard to the contact conditions on the lateral surface of the rod with the rollers, we first assume that the deformation of the directed curve satisfies (35). The condition for the lower lateral surface of the rod to contact the roller of radius R_1 is obtained by setting $\theta^1 = -h$ in (2), and the assistance of (16) and (26):

$$\begin{aligned} R_1\sin(\varphi_1)-O_1 &= \zeta+u_3-h\bar{\delta}_{13}, \\ -R_1(1-\cos(\varphi_1))-H_1 &= u_1-h-h\bar{\delta}_{11}, \end{aligned} \tag{43}$$

where $\varphi_1 = \varphi_1(\zeta)$, and O_1 and H_1+R_1 are the horizontal and vertical distances, respectively, from the origin to the center of the roller (see Fig. 2). Similarly, the contact condition for the upper lateral surface of the rod which contacts the roller of radius R_2 is

$$\begin{aligned} R_2\sin(\varphi_2)+O_2 &= \zeta+u_3+h\bar{\delta}_{13}, \\ R_2(1-\cos(\varphi_2))+H_2 &= u_1+h+h\bar{\delta}_{11}. \end{aligned} \tag{44}$$

We note that the relations (43) and (44) are independent of the nature of the contact.

It is convenient to obtain the following relations by differentiating (43) and (44) with respect to ζ:

$$\begin{aligned} R_1\frac{d\varphi_1}{d\zeta} &= \left(1+\frac{du_3}{d\zeta}-h\frac{d\bar{\delta}_{13}}{d\zeta}\right)\cos(\varphi_1)-\left(\frac{du_1}{\partial\zeta}-h\frac{d\bar{\delta}_{11}}{d\zeta}\right)\sin(\varphi_1), \\ R_2\frac{d\varphi_2}{d\zeta} &= \left(1+\frac{du_3}{d\zeta}+h\frac{d\bar{\delta}_{13}}{d\zeta}\right)\cos(\varphi_2)+\left(\frac{du_1}{d\zeta}+h\frac{\partial\bar{\delta}_{11}}{d\zeta}\right)\sin(\varphi_2). \end{aligned} \tag{45}$$

Using (43)–(45) in a similar manner, the corresponding relations for $d^2\varphi_\sigma/d\zeta^2$ may be obtained. When the contact between the rod and the roller is one of pure adhesion, by equating expressions for the velocity of points on the lateral surface of the rod and those of the roller, and then using (45), it may be shown that

$$\frac{d\varphi_\sigma}{d\zeta} = \frac{\Omega_\sigma}{c}. \tag{46}$$

In (46), Ω_1 and $-\Omega_2$ are the constant (clockwise) rotational speeds of the lower roller and upper roller, respectively. After integrating (46) and using the initial conditions $\varphi_\sigma(\zeta = 0) = \varphi_{\sigma 0}$, we obtain

$$\varphi_1 = \varphi_1(\zeta) = \frac{\Omega_1}{c}(\zeta-\zeta_{10}),\ \varphi_2 = \varphi_2(\zeta) = \frac{\Omega_2}{c}(\zeta-\zeta_{20}), \tag{47}$$

where $u_3(\zeta_{\sigma 0})+\zeta_{\sigma 0} = O_\sigma$.

When the rod is in contact with a roller, the contact forces exerted by the individual rollers on the rod are assumed to be uniformly distributed along its width or $\mathbf{E}_2$ direction. The force per unit length exerted by the lower roller on the rod is

$$q_{1t}(\cos(\varphi_1)\mathbf{E}_3-\sin(\varphi_1)\mathbf{E}_1)+q_{1n}(\sin(\varphi_1)\mathbf{E}_3+\cos(\varphi_1)\mathbf{E}_1), \tag{48}$$

where $q_{1t} = 0$ when frictionless slip is assumed, and q_{1n} and q_{1t} are independent when the adhesion case is considered. Similarly, when the rod is in contact with the upper roller, the force exerted on the rod is

$$q_{2t}(\cos(\varphi_2)\mathbf{E}_3+\sin(\varphi_2)\mathbf{E}_1)+q_{2n}(\sin(\varphi_2)\mathbf{E}_3-\cos(\varphi_2)\mathbf{E}_1). \tag{49}$$

The validity of the contact conditions, (43) and (44), requires that $q_{1n} > 0$, $q_{2n} > 0$, respectively. With the assistance of (16), (48) and (49), and relations from Naghdi (1982), the (non-trivial) assigned director forces, l_{11} and l_{13}, and the (non-trivial) assigned forces, f_1 and f_3 may be specified for the case when the rod is assumed to be in contact with both rollers:

$$\begin{bmatrix} \lambda f_3 \\ \lambda f_1 \\ \lambda l_{13}/h \\ \lambda l_{11}/h \end{bmatrix} = \begin{bmatrix} \cos(\varphi_1) & \sin(\varphi_1) & \cos(\varphi_2) & \sin(\varphi_2) \\ -\sin(\varphi_1) & \cos(\varphi_1) & \sin(\varphi_2) & -\cos(\varphi_2) \\ -\cos(\varphi_1) & -\sin(\varphi_1) & \cos(\varphi_2) & \sin(\varphi_2) \\ \sin(\varphi_1) & -\cos(\varphi_1) & \sin(\varphi_2) & -\cos(\varphi_2) \end{bmatrix} \begin{bmatrix} q_{1t} \\ q_{1n} \\ q_{2t} \\ q_{2n} \end{bmatrix} + \begin{bmatrix} 0 \\ -\lambda g \\ 0 \\ 0 \end{bmatrix}. \tag{50}$$

Clearly, the cases where the rod is in contact with only one roller or neither roller can be easily obtained from (50) by setting the appropriate elements of $\{q_{1t}, q_{2t}, q_{1n}, q_{2n}\}$ to 0.

As discussed in Section 4, the linear theory for this case reduces to four sets of equations. Two of these sets describe torsion, and flexure in the $\mathbf{E}_2$ direction (see Green *et al.* (1974b)). Using the formulation presented here these two sets of equations are trivially satisfied and are consequently omitted. The remaining two sets, (28), (29), (31) and (32), describe extension and flexure in the $\mathbf{E}_1$ direction. It will become evident that these two sets of equations will be coupled due to the contact conditions present in the problem of interest. At a point where the rod loses contact with one or both of the rollers, it is necessary to supplement the field equations with discontinuity conditions (20). As the contact forces are assumed to be finite, from (4) and (16), $\mathbf{F} = \mathbf{L}^\alpha = \mathbf{0}$. Due to the trivial satisfaction of two of the sets of equations, there are five non-trivial discontinuity conditions which need to be considered:

$$\left[\!\left[(\alpha_6-\lambda c^2)\frac{\mathrm{d}u_1}{\mathrm{d}\zeta}+\alpha_6\bar{\delta}_{13}\right]\!\right]=0,\quad \left[\!\left[(\alpha_{10}-\lambda y^{11}c^2)\frac{\mathrm{d}\bar{\delta}_{11}}{\mathrm{d}\zeta}\right]\!\right]=0,$$
$$\left[\!\left[(\alpha_1-\lambda c^2)\frac{\mathrm{d}u_3}{\mathrm{d}\zeta}+\alpha_7(\bar{\delta}_{11}+\bar{\delta}_{22})\right]\!\right]=0,$$
$$\left[\!\left[(\alpha_{16}-\lambda y^{11}c^2)\frac{\mathrm{d}\bar{\delta}_{13}}{\mathrm{d}\zeta}\right]\!\right]=0,\quad \left[\!\left[(\alpha_{11}-\lambda y^{22}c^2)\frac{\mathrm{d}\bar{\delta}_{22}}{\mathrm{d}\zeta}\right]\!\right]=0. \tag{51}$$

For completeness, it is necessary to supplement these five conditions with the stipulations that $\mathbf{u}$ and $\boldsymbol{\delta}_\alpha$ are continuous:†

$$[\![u_1]\!]=0, [\![u_3]\!]=0, [\![\bar{\delta}_{13}]\!]=0, [\![\bar{\delta}_{11}]\!]=0, [\![\bar{\delta}_{22}]\!]=0. \tag{52}$$

6.2. *Coupling between extensional and flexural deformations*

The conditions, (43) and (44), which hold when the rod contacts the rollers imply that the extensional $(u_3, \bar{\delta}_{11}, \bar{\delta}_{22})$ and flexural $(u_1, \bar{\delta}_{13})$ deformations are coupled, in general. The physics behind this coupling parallels, to a certain extent, the static case considered by Naghdi and Rubin (1989). They noted that it is caused by the difference in the normal forces on the upper and lower lateral surfaces of the beam, which in turn produces a flexural deformation.‡ In the case of steadily moving rods, this coupling is arguably more prevalent due to the profuse occurrence of contact mechanisms in applications. If the rod is in contact with the upper and lower rollers, then (43) and (44) are easily manipulated to show that the extensional and flexural deformations are coupled if there is any asymmetry in the

† Related conditions were recorded by Naghdi and Rubin (1989).
‡ Related remarks were also made by Essenburg (1975) and Love (1944), among others.

contact conditions. In particular, gravity, offsets in the horizontal and vertical positions of the rollers, differing roller diameters, or differing roller speeds, couple the balance laws for these two types of deformation. In other words, they all result in a difference between the contact forces on the upper and lower lateral surfaces of the rod.

It is illuminating to consider a simple case where the rod is only in contact with the lower roller over a region $\zeta \in (-L_3, L_4)$. The lengths L_3 and L_4 are not known *a priori* and the contact of the rod and the roller is assumed to be frictionless. In contrast to the adhesion situation, there is only one unknown contact force q_{1n}. Consequently, the assigned forces and assigned director forces are related (cf. (50)):

$$\lambda f_3 = q_{1n}\sin(\varphi_1), \lambda f_1 = q_{1n}\cos(\varphi_1) - \lambda g,$$
$$\lambda l_{13} = -hq_{1n}\sin(\varphi_1) = -h\lambda f_3, \lambda l_{11} = -hq_{1n}\cos(\varphi_1) = -h(\lambda f_1 + \lambda g). \tag{53}$$

In addition, the contact condition (43) serves to relate certain kinematical variables:

$$\bar{\delta}_{13} = \frac{1}{h}(\zeta + u_3 + O_1 - R_1\sin(\varphi_1)),$$
$$u_1 = -R_1(1 - \cos(\varphi_1)) - H_1 + h + h\bar{\delta}_{11}. \tag{54}$$

It should be clear from (54) that the extensional and flexural deformations are coupled. The equation $(28)_1$ determines q_{1n}:

$$q_{1n} = -\left((\alpha_6 - \lambda c^2)\frac{d^2 u_1}{d\zeta^2} + \alpha_6\bar{\delta}_{13} - \lambda g\right)\sec(\varphi_1). \tag{55}$$

The four remaining functions, $u_3(\zeta)$, $\bar{\delta}_{11}(\zeta)$, $\bar{\delta}_{22}(\zeta)$ and $\varphi(\zeta)$, are determined from $(28)_2$, (29), (31), (32), (53)–(55). It should be noted that $(28)_2$ and (31) are coupled both by the presence of q_{1n} in the assigned forces and assigned director forces, and (54). Their solution provides the contact force from (55) and the vanishing of this force determines L_3 and L_4. Finally, the eight constants of integration are determined by using (51) and (52) at the points $\zeta = -L_3$ and $\zeta = L_4$, the solution outside the contact region and its boundary conditions.

6.3. *The extensional deformation: an example*

We now consider the case where the rod is being pulled or drawn between two identical rollers with zero horizontal offsets (i.e., $O_\alpha = 0$). In addition, the gravitational force is ignored. Presuming appropriate boundary conditions, and $l_{13} = f_1 = 0$, the flexural equations will be trivially satisfied throughout the rod's length. These assumptions also imply that $q_{1n} = q_{2n}$ and $q_{1t} = q_{2t}$. The contact between the rollers and rod is assumed to be one of complete adhesion. Referring to the solution discussed in Appendix B, if we set $\bar{B} = 0$ in (B7), then the contact region will be symmetric about the origin and $q_{\alpha t}(\zeta)$ will be odd functions of ζ. Consequently, the $\mathbf{E}_2$ component of the resultant external moment relative to the center of each of the rollers is 0.

For the example shown in Fig. 3, the parameters are chosen to be:

$$h = w, \quad R = 20w, \quad H = 0.999w, \quad R\Omega = c,$$
$$\bar{A} = 0.002, \quad \bar{B} = 0, \quad \nu = 0.3. \tag{56}$$

The selection of $\bar{A}$ was governed by the requirement that the resulting deformation and forces are restricted to agree with the smallness assumptions of a linear theory (cf. (27)). This constant will determine the resultant forces at the boundaries of the contact region. In turn, this provides the resultant forces needed to pull the rod through the rollers. It is interesting to note that variations, by factors of 100, of the non-dimensional speed $\rho_0 c^2/E$ from 0.00025 did not prove to be a major influence on the results shown in Fig. 3.† Further,

† These figures were obtained with the assistance of the symbolic manipulation package Mathematica®.

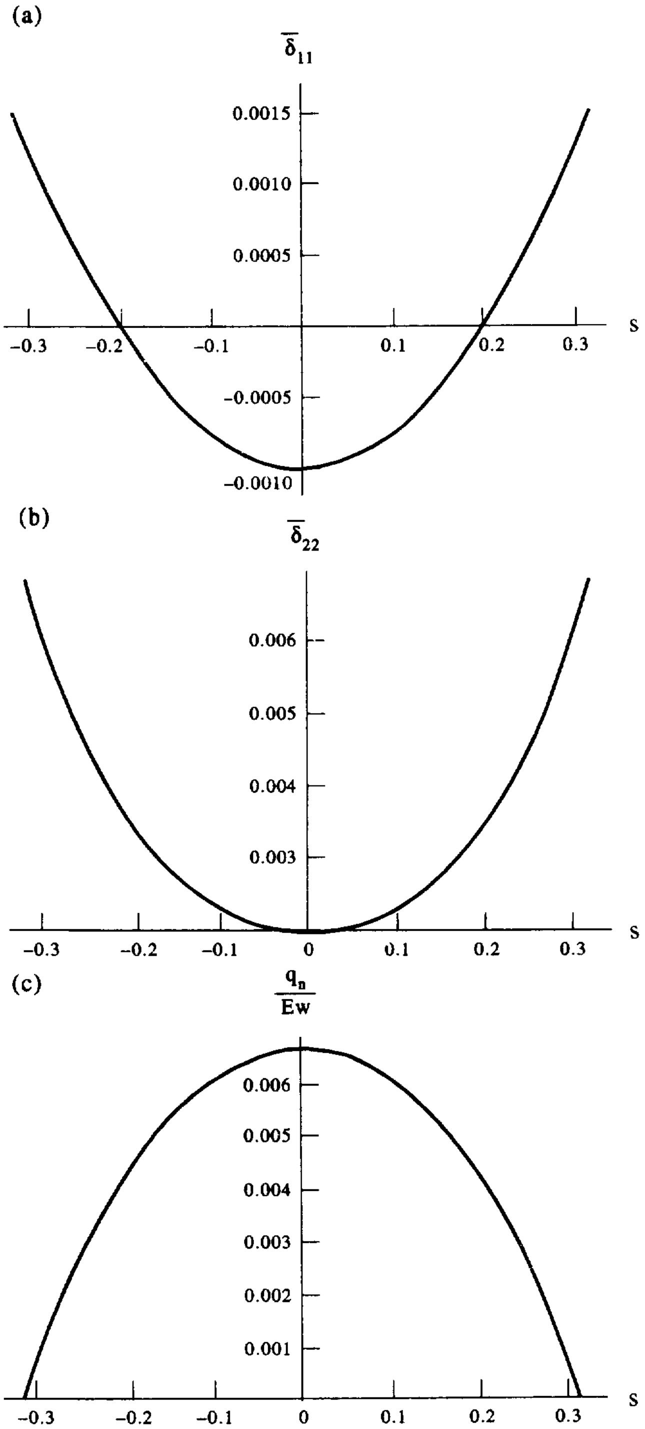

Fig. 3. Numerical results for the problem of a rod whose lateral surface is in complete adhesion with two identical rollers for $\rho_0 c^2/E = 0.00025$. The figures indicate (a) $\bar{\delta}_{11}(s)$, (b) $\bar{\delta}_{22}(s)$, (c) $q_t(s)/Ew$, (d) $q_n(s)/Ew$, and (e) $\mu(s) = |q_t|/q_n$, where $s = (R/w)\sin(\Omega\zeta/c)$ is the dimensionless arc-length parameter of the deformed material curve. The remaining parameter values are given by eqn (56).

the distribution of q_n is qualitatively in agreement with the results for a related problem of Bentall and Johnson (1967, 1968). These authors examined plane strain solutions of the linear theory of elasticity, in contrast to the results we present here.

As in the work of Bentall and Johnson (1967, 1968), it is evident from Fig. 3(c), (d) that an infinite coefficient μ of static Coulomb friction is required at the edge of the contact region for the adhesion assumption to apply. Further, the deformation of the rod is peculiar in that $\bar{\delta}_{11}$ becomes positive away from the center of the contact region. This occurs even

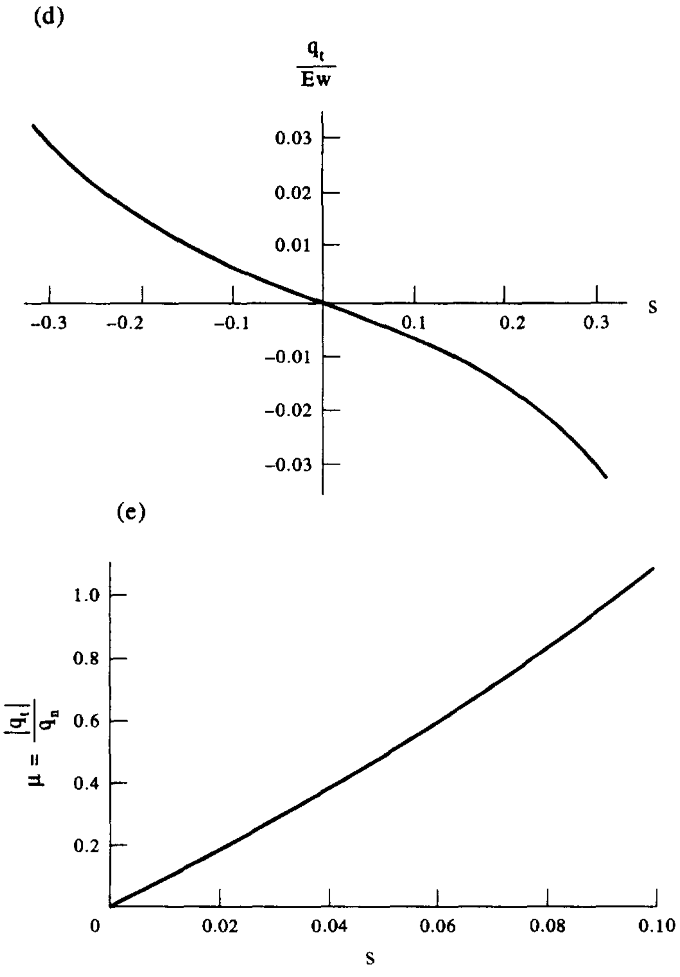

Fig. 3.—*Continued.*

though the rod is being compressed near $\zeta = 0$. In other words, the unlimited Coulomb friction available allows the rollers to "pull" the rod-like body onto them. A physically more realistic situation is to assume a finite μ. The resulting contact length may be obtained from Fig. 3(e). For a value of $\mu = 0.2$, this contact length is $0.042w$, a value which is in marked contrast to the contact length of $0.632w$ obtained when unlimited friction is available. Outside of this region the rod will either slip on, or lose contact with, the rollers. The behavior of the rod in the former case is obtained by paralleling, with some modifications to include dynamic Coulomb friction, the analysis discussed in the final paragraph of Appendix B. In the latter case, the behavior of the rod outside the contact region may be obtained using the results of Appendix A, (51) and (52).

7. CLOSING COMMENTS ON RELATED PROBLEMS

The previous two sections give partial results for many applications, and it is appropriate to recall here the assumptions used in Sections 5 and 6. These assumptions are that the deformation satisfies (35), and that the boundary conditions, assigned force **f** and assigned director forces $\mathbf{l}^{\alpha}$ are such that the set of torsional equations and one of the set of flexural equations are trivially satisfied. In applications these assumptions may not be realized, particularly when the rod-like body is moving through several sets of rollers and over several obstacles. For these cases the solution procedure for the full set of equations of the general linear theory is easily inferred from the discussions of Sections 5 and 6.

The solution methods discussed in Sections 5 and 6 are analogous to those used to solve problems of constrained motions in dynamics. In this respect, the contact conditions and contact forces are analogous to kinematical constraints and constraint forces, respectively. As is well known in dynamics, the chief difficulty in solving such problems is to decouple the equations for the unconstrained motion from those for the constraint forces. In dynamics, this involves a judicious choice of kinematical variables (or coordinate transformations) to describe the dynamical system. Once this has been performed, the resulting equations may be solved using numerical simulations or, in certain cases, analytically. For the applications of interest in this paper, we feel that a program of research, analogous to that outlined above for dynamical systems, is feasible. The results of this research would provide information on the deformation and stresses in rod-like bodies which appear in many applications. The work would also provide an attractive compliment to existing research in contact mechanics which uses other methods.

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APPENDIX A: ANALYTICAL RESULTS FOR THE SYSTEM DISCUSSED IN SECTION 5

Assuming that the motion is steady, the general solutions of the flexure equations, (28), (29), (33), (34) and (36), are†

$$\bar{\delta}_{13}(\zeta) = \bar{A}\sin(\omega_1\zeta) + \bar{B}\cos(\omega_1\zeta) + \frac{g}{c^2}\zeta + \bar{C},$$

$$u_1(\zeta) = \frac{\alpha_6}{\omega_1(\alpha_6 - \lambda c^2)}(\bar{A}\cos(\omega_1\zeta) - \bar{B}\sin(\omega_1\zeta)) - \frac{g}{2c^2}\zeta^2 - \bar{C}\zeta - \bar{D}, \tag{A1}$$

where $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are constants and

$$\omega_1 = \sqrt{\frac{\alpha_6\lambda c^2}{(\alpha_{16} - \lambda y^{11}c^2)(\alpha_6 - \lambda c^2)}}. \tag{A2}$$

It follows, from (29) and (A1), that

$$n_1(\zeta) = -\frac{\alpha_6\lambda c^2}{(\alpha_6 - \lambda c^2)}(\bar{A}\sin(\omega_1\zeta) + \bar{B}\cos(\omega_1\zeta)),$$

$$m_{13}(\zeta) = \alpha_{16}\left(\omega_1\bar{A}\cos(\omega_1\zeta) - \omega_1\bar{B}\sin(\omega_1\zeta) + \frac{\lambda g}{\lambda c^2}\right). \tag{A3}$$

The general solution for steady motions of the extensional equations, (31)–(34) and (36), are now summarized. For the director displacements, we obtain

$$\bar{\delta}_{11}(\zeta) = \bar{\delta}_{11}^{c1}(\zeta) + \bar{\delta}_{11}^{c2}(\zeta) - \bar{E}, \bar{\delta}_{22}(\zeta) = \bar{\delta}_{22}^{c1}(\zeta) + \bar{\delta}_{22}^{c2}(\zeta) - \bar{E}, \tag{A4}$$

where $\bar{E}$ is a constant. The remaining functions in (A4) are

$$\bar{\delta}_{11}^{c1}(\zeta) = \bar{F}\sinh(\omega_2\zeta) + \bar{G}\cosh(\omega_2\zeta),$$

$$\bar{\delta}_{11}^{c2}(\zeta) = \bar{H}\sinh(\omega_3\zeta) + \bar{I}\cosh(\omega_3\zeta),$$

$$\bar{\delta}_{22}^{c1}(\zeta) = \beta_3\bar{\delta}_{11}^{c1}(\zeta), \bar{\delta}_{22}^{c2}(\zeta) = \beta_4\bar{\delta}_{11}^{c2}(\zeta). \tag{A5}$$

where $\bar{F}$, $\bar{G}$, $\bar{H}$, $\bar{I}$ are constants,

$$\omega_{2,3}^2 = \beta_1 \pm \sqrt{\beta_1^2 - \beta_2} \tag{A6}$$

† In this appendix, it is assumed that $c \neq 0$ and the temporary use of the symbols $\bar{A}$ and $\bar{B}$ here should not be confused with those used in Section 6 or Appendix B. It should be noted that the corresponding solutions for the case $c = 0$ cannot be obtained by taking the limiting value of the solutions presented here. For convenience, the identities $(33)_{1,3}$ are also used in this appendix.

and

$$\beta_1 = \frac{1}{2}\left(\alpha_1 - \frac{\alpha_7^2}{\alpha_1 - \lambda c^2}\right)\left(\frac{1}{\alpha_{10} - \lambda y^{11} c^2} + \frac{1}{\alpha_{11} - \lambda y^{22} c^2}\right),$$

$$\beta_2 = \left(\alpha_1 + \alpha_7 - \frac{2\alpha_7^2}{\alpha_1 - \lambda c^2}\right)\left(\frac{\alpha_1 - \alpha_7}{(\alpha_{10} - \lambda y^{11} c^2)(\alpha_{11} - \lambda y^{22} c^2)}\right),$$

$$\beta_{2+\sigma} = \frac{\alpha_7^2 + (\alpha_1 - \lambda c^2)((\alpha_{10} - \lambda y^{11} c^2)\omega_{\sigma+1}^2 - \alpha_1)}{\alpha_7(\alpha_1 - \lambda c^2 - \alpha_7)}, \sigma = 1, 2. \tag{A7}$$

For the displacement $u_3(\zeta)$, we obtain

$$u_3(\zeta) = \bar{J} + \bar{E}\left(\frac{\alpha_1}{\alpha_7} + 1\right)\zeta - \beta_5(\bar{F}\cosh(\omega_2\zeta) + \bar{G}\sinh(\omega_2\zeta)) - \beta_6(\bar{H}\cosh(\omega_3\zeta) + \bar{I}\sinh(\omega_3\zeta)), \tag{A8}$$

where $\bar{J}$ is a constant and

$$\beta_{4+\sigma} = \frac{(\alpha_{10} - \lambda y^{11} c^2)\omega_{\sigma+1}^2 + \alpha_7 - \alpha_1}{\omega_{\sigma+1}(\alpha_1 - \lambda c^2 - \alpha_7)}, \sigma = 1, 2. \tag{A9}$$

With the assistance of (32), expressions for the following fields may be obtained from (A4), (A8) and (A9):

$$n_3(\zeta) = \left(\frac{\alpha_1^2 + \alpha_1\alpha_7 - 2\alpha_7^2}{\alpha_7}\right)\bar{E} + \sum_{\sigma=1}^{2} (\alpha_7(1 + \beta_{(2+\sigma)}) - \alpha_1\omega_{(1+\sigma)}\beta_{(4+\sigma)})\bar{\delta}_{11}^{\sigma}(\zeta),$$

$$m_{11}(\zeta) = \alpha_{10}(\omega_2(\bar{F}\cosh(\omega_2\zeta) + \bar{G}\sinh(\omega_2\zeta)) + \omega_3(\bar{H}\cosh(\omega_3\zeta) + \bar{I}\sinh(\omega_3\zeta))),$$

$$m_{22}(\zeta) = \alpha_{11}(\beta_3\omega_2(\bar{F}\cosh(\omega_2\zeta) + \bar{G}\sinh(\omega_2\zeta)) + \beta_4\omega_3(\bar{H}\cosh(\omega_3\zeta) + \bar{I}\sinh(\omega_3\zeta))). \tag{A10}$$

For the two regions of the rod, upstream and downstream of the obstacle, we obtain general solutions of the form (A1)–(A10). The constants for the upstream region are designated by a superscript I, (e.g., $\bar{A}^{\rm I}$, $\bar{B}^{\rm I}$), and those for the downstream region are designated by a superscript II (e.g., $\bar{A}^{\rm II}$, $\bar{B}^{\rm II}$). Excluding $\bar{q}$, there are a total of twenty unknown constants. The two sets of solutions are related by the contact, discontinuity, and continuity conditions at $\zeta = 0$ (i.e., (37), (38) and (39)) which serve to reduce the number of unknowns by ten. Specifically, several identities are obtained after some manipulations:

$$\bar{A}^{\rm I} = \bar{A}^{\rm II} \equiv \bar{A}, \bar{D}^{\rm I} = \bar{D}^{\rm II} \equiv \bar{D}, \bar{E}^{\rm I} = \bar{E}^{\rm II} \equiv \bar{E}, \bar{G}^{\rm I} = \bar{G}^{\rm II} \equiv \bar{G}, \bar{I}^{\rm I} = \bar{I}^{\rm II} \equiv \bar{I},$$

$$\bar{B}^{\rm II} = \bar{B}^{\rm I} + \frac{\bar{q}}{\lambda c^2}, \bar{C}^{\rm II} = \bar{C}^{\rm I} - \frac{\bar{q}}{\lambda c^2},$$

$$\bar{F}^{\rm II} = \bar{F}^{\rm I} + \frac{\beta_4\bar{q}h}{(\beta_4 - \beta_3)(\alpha_{10} - \lambda y^{11} c^2)\omega_2},$$

$$\bar{H}^{\rm II} = \bar{H}^{\rm I} - \frac{\beta_3\bar{q}h}{(\beta_4 - \beta_3)(\alpha_{10} - \lambda y^{11} c^2)\omega_3},$$

$$\bar{J}^{\rm II} = \bar{J}^{\rm I} + \frac{(\beta_4\beta_5\omega_3 - \beta_3\beta_6\omega_2)\bar{q}h}{(\beta_4 - \beta_3)(\alpha_{10} - \lambda y^{11} c^2)\omega_2\omega_3}. \tag{A11}$$

With the assistance of (A11), we note that there are a total of eleven (independent) unknowns in the boundary value problem. These may be determined using ten boundary conditions, (e.g., (41) and (42)), and the contact condition (37):

$$\left(\sqrt{\frac{\alpha_6(\alpha_{16} - \lambda y^{11} c^2)}{\lambda c^2(\alpha_6 - \lambda c^2)}}\right)\bar{A} - \bar{D} - h(\bar{I} + \bar{G} - \bar{E}) = L, \tag{A12}$$

where, in writing (A12), the identifies (A2) and $(\mathrm{A11})_{1,5}$ have been invoked.

APPENDIX B: ANALYTICAL RESULTS FOR THE SYSTEM DISCUSSED IN SECTION 6

Here we consider the case where adhesion is assumed and the parameters of the system are

$$g = 0, R_\sigma = R, O_\sigma = 0, H_\sigma = H, \Omega_\sigma = \Omega, \zeta_{\sigma 0} = 0. \tag{B1}$$

The rod is in contact with both rollers over a region $\zeta \in (L_1, L_2)$, where $-L_1, L_2 > 0$ and are not known *a priori*. For this case, (43), (44) and (47) may be used to determine $u_1(\zeta)$, $u_3(\zeta)$, $\bar{\delta}_{11}(\zeta)$ and $\bar{\delta}_{13}(\zeta)$:

$$u_1(\zeta) = 0, u_3(\zeta) = -\zeta + R\sin\left(\frac{\Omega\zeta}{c}\right),$$

$$\bar{\delta}_{11}(\zeta) = \frac{1}{h}\left(-R\cos\left(\frac{\Omega\zeta}{c}\right) + R + H - h\right), \bar{\delta}_{13}(\zeta) = 0. \tag{B2}$$

Before the balance laws, (28) and $(31)_{1,2}$, and (B2) are used to determine the four independent contact forces, $\bar{\delta}_{22}(\zeta)$ must be determined from $(31)_3$:

$$(\alpha_{11} - \lambda y^{22}c^2)\frac{d^2\bar{\delta}_{22}}{d\zeta^2} - \alpha_2\bar{\delta}_{22} = \frac{\alpha_7}{h}(R+H-h) - \alpha_9 - \left(\frac{\alpha_7}{h} - \frac{\alpha_9\Omega}{c}\right)R\cos\left(\frac{\Omega\zeta}{c}\right), \tag{B3}$$

where we have used the constitutive relations, (32), (33) and (34), and the functions $(B2)_{2,3}$. The equation (B3) is solved using standard methods:

$$\bar{\delta}_{22}(\zeta) = \bar{A}\cosh(\omega\zeta) + \bar{B}\sinh(\omega\zeta) + \bar{\delta}^p_{22}(\zeta), \tag{B4}$$

where $\bar{A}$ and $\bar{B}$ are constants, and

$$\bar{\delta}^p_{22}(\zeta) = \frac{\alpha_9}{\alpha_2} - \frac{\alpha_7}{\alpha_2}\left(\frac{(R+H-h)}{h}\right) + \frac{\left(\alpha_7\dfrac{R}{h} - \alpha_9\dfrac{\Omega R}{c}\right)}{(\alpha_{11} - \lambda y^{22}c^2)\left(\dfrac{\Omega}{c}\right)^2 + \alpha_2}\cos\left(\frac{\Omega\zeta}{c}\right),$$

$$\omega^2 = \frac{\alpha_2}{\alpha_{11} - \lambda y^{22}c^2}. \tag{B5}$$

The contact forces are obtained using (B1)–(B3) and (28)–(34):

$$2q_t = 2q_{1t} = 2q_{2t} = \lambda f_3\cos\left(\frac{\Omega\zeta}{c}\right) + \frac{1}{h}\lambda l_{11}\sin\left(\frac{\Omega\zeta}{c}\right),$$

$$2q_n = 2q_{1n} = 2q_{2n} = \lambda f_3\sin\left(\frac{\Omega\zeta}{c}\right) - \frac{1}{h}\lambda l_{11}\cos\left(\frac{\Omega\zeta}{c}\right), \tag{B6}$$

where

$$\lambda f_3 = -\alpha_9\omega(\bar{A}\sinh(\omega\zeta) + \bar{B}\cosh(\omega\zeta)) + \left[(\alpha_3 - \lambda c^2)\left(\frac{R\Omega}{c}\right) - \alpha_8\frac{R}{h} + \frac{\alpha_9\left(\alpha_7\dfrac{R}{h} - \alpha_9\dfrac{R\Omega}{c}\right)}{\alpha_2 + (\alpha_{11} - \lambda y^{22}c^2)\left(\dfrac{\Omega}{c}\right)^2}\right]\left(\frac{\Omega}{c}\right)\sin\left(\frac{\Omega\zeta}{c}\right),$$

$$\lambda l_{11} = \alpha_7(\bar{A}\cosh(\omega\zeta) + \bar{B}\sinh(\omega\zeta)) + \left(\frac{\alpha_7^2}{\alpha_2} - \alpha_1\right)\left(1 - \frac{R}{h} - \frac{H}{h}\right) + \alpha_7\frac{\alpha_9}{\alpha_2} - \alpha_8 + \left[\alpha_8\left(\frac{R\Omega}{c}\right) + \frac{\alpha_7\left(\alpha_7\dfrac{R}{h} - \alpha_9\dfrac{R\Omega}{c}\right)}{\alpha_2 + (\alpha_{11} - \lambda y^{22}c^2)\left(\dfrac{\Omega}{c}\right)^2} - \left(\frac{R}{h}\right)\left(\alpha_1 + \left(\frac{\Omega}{c}\right)^2(\alpha_{10} - \lambda y^{11}c^2)\right)\right]\cos\left(\frac{\Omega\zeta}{c}\right). \tag{B7}$$

The contact length parameters, L_1 and L_2 may be determined using $(B6)_2$. It is evident from the expressions for q_n and q_t that they may only be determined using numerical methods. The arc-length of the deformed material curve for these cases is obtained from $(26)_1$ and (B2).